Compressed Sensing

Solt Kovács & Yoann Trellu

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ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Initially...

Sensing a signal and *compressing* it are two distinct processes.

Example (Selfie as a signal)

We take a picture using a sensor for each pixels (ccds, name of the sensing device), and then compress it using the JPEG standard.

Example (Song as a signal)

Using a microphone, we record the audio signal (diaphragm), and then compress it using the MP3 standard.

$$\underbrace{\tilde{x} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \tilde{x}}_{\textit{Sensing}} \implies \underbrace{x = \Psi \tilde{x}}_{\textit{Compression}} \implies ☺ \implies \underbrace{\tilde{x} = \Psi^{-1} x}_{\textit{Decompression}}.$$

But why not merge the first two steps ?

$$\underbrace{\tilde{y} = \Phi \tilde{x}}_{Compressed\ Sensing} \implies ☺ \implies \underbrace{\tilde{x} = f(\tilde{y}, \Phi, \Psi)}_{Decompression}.$$

- Φ is going to be a specifically designed *sensing* matrix
- f is going to rely on a L_1 minimization.
- We will assume that $\tilde{x}$ can be written in a sparse way (or at least compressible) in the Ψ basis.

Exact Recovery of Sparse Signals

Let us sense a sparse signal $x \in \mathbb{R}^N$ with matrix $\Phi \in \mathbb{R}^{N \times N}$. We get measurements

$$y := \Phi x.$$

As long as Φ obeys a *Restricted Isometry Property*, solving

$$x^\star = \text{argmin} ||z||_{l_1} \qquad \text{subject to} \qquad \Phi z = y$$

exactly recovers the signal x (i.e. $x = x^\star$).

Stable Recovery from imperfect measurements

In reality, noise is introduced and signals won't be exactly sparse. With ϵ a bound on the noise level, the problem is formulated as:

$$x^\star = \operatorname{argmin}||z||_{l_1} \qquad \text{subject to} \quad ||\Phi z - y||_{l_2} \leq \epsilon. \qquad (1)$$

Theorem

Take x an arbitrary vector in $\mathbb{R}^N$ and let x_S be the truncated vector corresponding to the S largest absolute values of x. Then under some assumptions on Φ, the solution $x^\star$ to equation 1 obeys

$$||x^\star - x||_{l_2} = C_{1,S}\epsilon + C_{2,S}\frac{||x - x_S||_{l_1}}{\sqrt{S}}.$$

Lensless Camera, Bell Labs, 2013

Example where the two processes are merged.

It has a simple architecture:

- one light sensor
- an aperture assembly
- M measurements

As a result the signal is already recorded in compressed format.

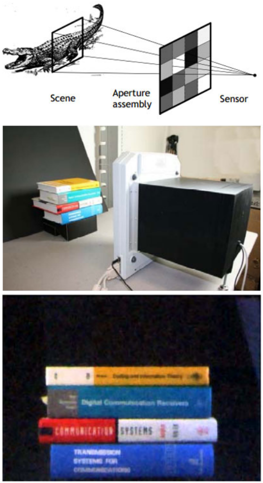

Figure: Experiment Sketch, and image taken with 25% measurements

Sensing Matrix Design

Let us now talk about the design of the sensing matrix Φ.

Restricted Isometry Property

Definition

A matrix Φ satisfies the restricted isometry property of order K if there exists a $\delta_K \in (0,1)$ such that

$$(1-\delta_K)||x||_2^2 \leq ||\Phi x||_2^2 \leq (1+\delta_K)||x||_2^2$$

holds for all $x \in \Sigma_K = \{x : ||x||_0 \leq K\}$.

Remark

If for the sensing matrix Φ the restricted isometry property of order $3S$ and $4S$ is fulfilled and in addition $\delta_{3S} + \delta_{4S} < 2$, then the previously mentioned theorem holds.

Main Theorem

Theorem

Fix $\delta_K \in (0,1)$. *Let* Φ *be an* $M \times N$ *random matrix whose entries* Φ_{ij} *are i.i.d* $\sim SSG(1/M)$. *If*

$$M \geq \kappa_1 K \log(N/K),$$

then Φ *satisfies the RIP of order* K *with the prescribed* δ_K *with probability exceeding* $1 - 2e^{-\kappa_2 M}$, *where* κ_1 *is arbitrary and* $\kappa_2 = \delta^2/2\kappa^* - \log(42e/\delta_K)/\kappa_1$.

Sub-Gaussians

Definition

A random variable X is called sub-Gaussian, denoted $X \sim SG(c^2)$ if there $\exists c > 0$ s.t.

$$E[e^{tX}] \leq \exp(c^2 t^2/2) \; \forall t > 0.$$

If the above inequality is satisfied for $c^2 = \sigma^2 = E[X^2]$, then we call X strictly sub-Gaussian, denoted $X \sim SSG(\sigma^2)$.

Example

- $X \sim N(0, \sigma^2)$, then $X \sim SSG(\sigma^2)$.
- $X : E[X] = 0$ and $P(|X| \leq B) = 1$ for some B, then $X \sim SG(B^2)$.

Linear combinations of indep. *SSG*-s

Lemma

Let $X_1, \ldots, X_n$ be i.i.d. and $\alpha \in \mathbb{R}^n$.
If $X \sim SG(c^2)$, then $\sum_{i=1}^n \alpha_i X_i \sim SG(c^2||\alpha||^2)$.
If $X \sim SSG(\sigma^2)$, then $\sum_{i=1}^n \alpha_i X_i \sim SSG(\sigma^2||\alpha||^2)$.

Proof.

$$E[\exp(t\sum_{i=1}^n \alpha_i X_i)] = E[\prod_{i=1}^n \exp(t\alpha_i X_i)] = \prod_{i=1}^n E[\exp(t\alpha_i X_i)]$$

$$\leq \prod_{i=1}^n \exp(c^2(t\alpha_i)^2/2) = \exp((\sum_{i=1}^n \alpha_i^2)c^2t^2/2).$$

For the strictly sub-Gaussian case we replace c with σ and use

$$E[(\sum_{i=1}^n \alpha_i X_i)^2] = \sum_{i=1}^n E[(\alpha_i X_i)^2] = \sum_{i=1}^n \alpha_i^2 E[X_i^2] = \sigma^2 \sum_{i=1}^n \alpha_i^2$$

Concentration of Measure

Theorem

Let $X = (X_1, X_2, ..., X_M)$ *be a vector with i.i.d.* $\sim SSG(\sigma^2)$ *entries. Then*

$$E[||X||_2^2] = M\sigma^2$$

and for any $\epsilon > 0$,

$$P(|||X||_2^2 - M\sigma^2| \geq \epsilon M\sigma^2) \leq 2\exp(\frac{-M\epsilon^2}{\kappa^*})$$

with $\kappa^* = 2/(1 - \log(2)) \approx 6.52$.

- Proof goes via Markov inequality.
- The (squared) norm of the vector concentrates around its expected value with exponentially high probability as M grows.
- Similar results hold for sub-Gaussian random variables as well, however, the bounds might not be made arbitrarily tight. This is the reason we use strictly sub-Gaussian random variables.

Corollary

Suppose that Φ is an $M \times N$ matrix whose entries are i.i.d $SSG(1/M)$ distributed. Let $Y = \Phi x$ for $x \in \mathbb{R}^N$. Then for any $\epsilon > 0$ and any $x \in \mathbb{R}^N$,

$$E[||Y||_2^2] = ||x||_2^2$$

and

$$P(|||Y||^2 - ||x||^2| \geq \epsilon ||x||^2) \leq 2\exp(\frac{-M\epsilon^2}{\kappa^*}) \qquad (2)$$

Proof.

Note $Y_i = \sum_{j=1}^N \Phi_{ij} x_j$. As Φ_{ij} are i.i.d., $Y_i \sim SSG(||x||^2/M)$. Thus, we can apply the previous theorem for the vector Y to get the result.

□

Points on unit balls

Lemma

Let $\epsilon \in (0,1)$ be given. There exists a set of points Q s.t.

$$||q||_2 = 1 \ \forall q \in Q \text{ and } |Q| \leq (3/\epsilon)^K,$$

and for any $x \in \mathbb{R}^K$ with $||x||_2 = 1$ there is a point $q \in Q$ satisfying $||x - q||^2 \leq \epsilon$.

Proof.

On the blackboard. □

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Proof:

The main idea is:

- reduce the problem to all K-sparse x with $||x||_2 = 1$, since Φ is linear
- to construct a set of points in each K-dimensional subspace (via lemma of the unit balls)
- apply the corollary (concentration of $||\Phi x||_2^2$) to all of these points through a union bound
- extend the result to all possible K-sparse signals

Summary:

What we have learned:

- Idea behind compressive sensing
- The (strictly) sub-Gaussian class of random variables
- Sensing matrix design using SSG random variables

What one could still look at:

- Where compressive sensing is used in reality, e.g. MRI scans
- Further theoretical results concerning compressive sensing
- Choices of basis
- Related results involving random matrices, e.g. the Johnson-Lindenstrauss Lemma

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