Semester Project
year 2012-2013

Numerical methods for SDEs and SDEs with jumps

Author:
Yoann Trellu
Supervisors:
Prof. Assyr Abdulle
Adrian Blumenthal

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

November 19, 2012

Contents

1 Introduction

2 Numerical method for ODEs

2.1 Cauchy Problem

The Cauchy problem is an ordinary differential equation (ODE) of order one with boundary conditions. Let I be an interval of $\mathbb{R}$ and $f(t, y)$ a given function defined on the product $S = I\times]-\infty, +\infty[$, continuous regarding t and y. Then the scalar form is written as follows:

$$\begin{cases} y' = f(t, y) & t \in I \\ y(t_0) = y_0 \end{cases} \tag{1}$$

We use several numerical methods to approach this problem. Well-known forward and backward euler methods use the fact that it is equivalent to write the problem in the integral form : $y(t) - y_0 = \int_{t_0}^{t_1} f(\tau, y(\tau))d\tau$.

2.2 The θ-method

We want to approximate the solution of the problem (1) using the θ-method. To do so, we first need to discretize the interval of time I. Let h be the distance $|t_{i+1} - t_i|$, where t_i are nodes forming a grid for our interval. We take $\theta \in [0, 1]$ as a free parameter to fix. Plus, because it offers more convenience, we shall use the notation $u_n = u(t_n)$ throughout this work. Then, it is written as

$$u_{n+1} = u_n + h(1-\theta)f(t_n, u_n) + \theta h f(t_{n+1}, u_{n+1}). \tag{2}$$

This method is motivated by a compromise between forward and backward Euler. Note that if $\theta = 0$ we have the forward Euler, with $\theta = 1$ the backward Euler, and if $\theta = 1/2$, the Crank-Nicolson method.

Numerical methods are characterized among others, by their order of convergence and their stability domain. The next two sections will give a good overview of how well the θ-method behave. Such a method is of particular interest when you can compute the most suitable θ related to your problem. We will see just below that you can even change the order of the method of a factor one with the right θ.

2.3 Order of Convergence

We try to investigate the error of the θ-method made after one step. To do so, we bound $|u_{n+1} - u_n^*|$ where $u_n^* = y_n + (1-\theta)hf(t_n, y_n) + \theta h f(t_{n+1}, y_{n+1})$ is one step of the θ-method starting from the exact given y_n. Assuming $y \in \mathcal{C}^4$, Taylor expansion around t_n yields

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2}y_n'' + \frac{h^3}{6}y_n''' + O(h^4).$$

Similarly we have

$$y_{n+1}' = y_n' + hy_n'' + \frac{h^2}{2}y_n''' + O(h^3).$$

Hence,

$$|y_{n+1} - u_n^*| = \left| y_n + hy_n' \frac{h^2}{2}y_n'' + \frac{h^3}{6}y_n''' + O(h^4) - y_n - (1-\theta)hy_n' - h\theta y_{n+1}' \right| =$$

$$= \left| \frac{h^2}{2}y_n'' + \frac{h^3}{6}y_n''' + O(h^4) + h\theta y_n' - h\theta \left(y_n' + hy_n'' + \frac{h^2}{2}y_n''' + \frac{h^3}{6}y_n^{(iv)} + O(h^4) \right) \right| =$$

$$= \left| h^2(\frac{1}{2} - \theta)y_n'' + h^3(\frac{1}{6} - \frac{\theta}{2})y_n''' + O(h^4) \right| = \epsilon_{n+1}.$$

Let $h\tau_{n+1}(h) = \epsilon_{n+1}$ and $\tau(h) = \max_{0 \le n \le N_h - 1} \tau_n(h)$. Recall that a method is of order p if $\tau(h) = O(h^p)$ for $h \to 0$. This implies

$$|y_{n+1} - u_n^*| \le \left(\frac{1}{2} - \theta\right) C_1 h^2 + \left(\frac{1}{6} - \frac{\theta}{2}\right) C_2 h^3$$

with C_1 and C_2 positive suitable constants. Thus, when $\theta = 1/2$, our scheme is of order two and of order one otherwise.

2.4 Stability

Consider the following Cauchy problem

$$\begin{cases} y' = \lambda y \\ y(0) = 1. \end{cases} \tag{3}$$

It can be shown that the exact solution to (3) is given by $y(t) = e^{\lambda t}$. We want to find the domain $S_{exact} = \{\lambda \in \mathbb{C} : \lim_{t \to \infty} |y(t)| \le c\}$ where c denotes a constant. Note that $|e^{\lambda t}| = e^{\Re(\lambda)t}$ remains bounded as t tends to infinity if and only if $\Re(\lambda) \le 0$. Thus we have $S_{exact} = \{\lambda \in \mathbb{C} | Re(\lambda) \le 0\}$. Equivalently, for numerical methods the following domain is of interest:

$$S_{approx} = \{h\lambda \in \mathbb{C} : (y_n)_{n \ge 0} \le c\}.$$

We will investigate stability of the θ-method by focusing on three cases, $\theta \in \{0, 1/2, 1\}$. Firstly, if $\theta = 0$, we go back to the explicit Euler method $u_{n+1} = u_n + h\lambda u_n$ for $n \ge 0$ and $u_0 = 1$. By induction we get $u_n = (1 + h\lambda)^n$, $n \ge 0$. Thus,

$$(u_n)_{n \ge 0} \text{ bounded } \iff |1 + h\lambda| \le 1 \iff h\lambda \in B(-1, 1).$$

B(x,y) denote an open ball in the complex plan with x as center and a radius of y. Second, when $\theta = 1/2$, we have $u_{n+1} = u_n + h(1-\theta)\lambda u_n + h\theta\lambda u_{n+1}$ for $n \ge 0$ and $u_0 = 1$. Applying the same thinking, we get $u_n = (1 + 1/2h\lambda)^n \times (1 - 1/2h\lambda)^{-n}$, $n \ge 0$.

$$(u_n)_{n \ge 0} \text{ bounded } \iff \frac{|1 + 1/2h\lambda|}{|(1 - 1/2h\lambda)|} \le 1 \iff h\lambda \in \{z \in \mathbb{C} | Re(z) \le 0\}$$

Finally, $\theta = 1$ gives implicit Euler method $u_{n+1} = u_n + h\lambda u_{n+1}$ for $n \ge 0$ and $u_0 = 1$. We compute $u_n = (1 - h\lambda)^{-n}$, $n \ge 0$ and conclude

$$(u_n)_{n \ge 0} \text{ bounded } \iff h\lambda \in \mathbb{C} \setminus B(1, 1).$$

A word can be added on $\mathcal{A}$-stability. A numerical method is said to be $\mathcal{A}$-stable if $S_{exact} \subset S_{approx}$. Hence, for $\theta \in \{1/2, 1\}$ (actually for $1/2 \le \theta \le 1$) our method is $\mathcal{A}$-stable.

Figure 1 show stability domains (S_{approx}) calculated above. (???)

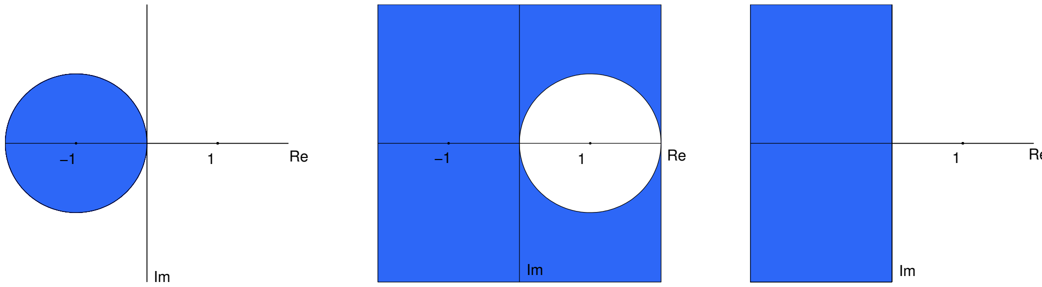

Figure 1: Complex plan with stability domains (light blue) for $\theta \in \{0, 1/2, 1\}$.

3 Numerical methods for SDEs

3.1 Brownian Motion

Some foundations and reminders need to be said in order to introduce properly the Brownian motion.

Firstly we will denote by $\mathcal{N}(\mu, \sigma^2)$ the normal (gaussian) distribution which has μ as mean, and σ^2 as variance. This low of probability is involved in a great variety of applications mainly due to the *Central Limit Theorem*. It is thus no surprise to take notice here of its use.

Secondly recall that a stochastic process is a random variable that depends on time. Properly speaking, we should enclose notions of *universe*, *fields* and other probabilistic notions. Thereafter we won't use such a formalism as our main interest is the numerical approximation of these objects.

Definition ([1])**.** *A standard Brownian motion (or standard Wiener process) over* $[0, T]$*, is a stochastic process which satisfies the following three conditions:*

1. *(Independence of increments) For* $0 \leq s < t < u < v \leq T$ *the increments* $W(t) - W(s)$ *and* $W(v) - W(u)$ *are independent.*
2. *(Gaussian increments) For* $0 \leq s < t \leq T$, $W(t) - W(s) \sim \sqrt{t-s}\,\mathcal{N}(0, 1)$
3. *(Continuity of paths)* $W(t)$, $t \geq 0$ *are continuous functions of* t*.*

Later on, we will consider Brownian motions such that $P(W(0) = 0) = 1$ (unless stated otherwise). Brownian motion was first developed in physics to explain trajectories of moving particles in a fluid. Because it would constantly be bombarded by the molecules of the fluid, its trajectory is presumably random in time. On this subject, stochastic theory introduces first *additive process* and the

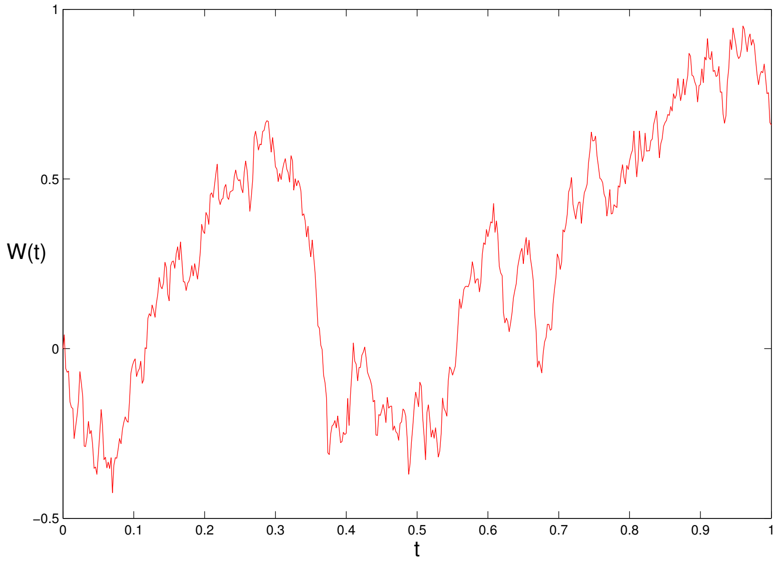

Figure 2: Discretized Brownian path in one dimension

Lévy Modification before the Brownian motion. Hence, the latter is also called a *Lévy process of Gauss type* (see [4]).

Figure 2 represents a Brownian motion over time. We can observe twitches going up and down; this comes from the discretization of the time. Each node t_i is normally distributed with mean t_{i-1} and fixed variance. That is, after each steps it will randomly goes either up or down but not too distant from the previous node.

3.2 Stochastic differential equations

An equation of the form

$$dX(t) = f(X(t), t)dt + g(X(t), t)dW(t),\ X(0) = X_0, \tag{4}$$

where $W(t)$ is a Brownian motion and functions $f(x,t)$, $g(x,t)$ are given, is called a stochastic differential equation (SDE). This is in fact merely a notation. The rigorous form is given by the integral

$$X(t) = X(0) + \int_0^t f(X(s), s)ds + \int_0^t g(X(s), s)dW(s). \tag{5}$$

The two functions f and g are called drift and diffusion term respectively. Because a Brownian Motion is nowhere differentiable [1], resolving this equation requires some stochastic tools. We will use a theoretical result known as Itô's formula but we won't go into further details. The main difficulty here lies in the definition of stochastic integrals. How can we define $\int_0^t X(t)dW(t)$? We want some properties to be verified: Firstly, if $X(t)$ is some constant c, then the integral should be $c(W(t)-W(0))$. Secondly, if $X(t)$ is piecewise constant, then we should have a sum of $c_i(W(s_{i+1}) - W(s_i))$ (the c_i would be the constant values taken in those intervals of time) as a result. This motivates the definition of what is called the Itô integral, named after Kiyosi Itô a Japanese mathematician

$$\int_0^T X(s)dW(t) = \lim_{n\to+\infty} \sum_{i=0}^{n-1} c_i(W(s_{i+1}) - W(s_i)).$$

The procedure reminds the construction of Lebesgues integrals; that is, approaching it by the infinite sum of simple functions. In numerical analysis, constants c_i are replaced by the function value at the node t_i $(h(t_i))$. Such a construction is called a *left-hand* sum, as we take this value at the left of the interval $[W(t_i), W(t_{i+1})]$. An alternative to the Itô integral is given by the Stratonovich integral. Instead of having a *left-hand* sum, we take the function value at the midpoint $h(t_i + t_{i+1})$ which gives rise to a *midpoint* sum. Some numerical approximations converges to the Stratonovich form besides, there exists transformations to go from one to the other.

Finally, if $X(t)$ satisfies (5), that means integrals are well defined with the second being an *Itô integral*. Naturally this leads us to make assumptions for f and g. Moreover, we take the opportunity to define a strong solution : (??) Simplifications on notations can be made here to ease the reading experience; we shall from now on write $f(X(t), t) = f(t)$. This means, we consider $f(t)$ as being dependent on t and the whole past of the processes $X(t)$ and $W(t)$.

Theorem 1 (Itô's formula for $f(X(t))$ [1])**.** *Let $X(t)$ have a stochastic differential for $0 \le t \le T$*

$$dX(t) = f(t)dt + g(t)dW(t). \tag{6}$$

If h(X) is C^2, then the stochastic differential of the process $Y(t) = h(X(t))$ exists and is given by

$$\begin{aligned} dh(X(t)) &= h'(X(t))dX(t) + \frac{1}{2}h''(X(t))g(t)^2dt \\ &= (h'(X(t))f(t) + \frac{1}{2}h''(X(t))g(t)^2)dt + h'(X(t))g(t)dW(t). \end{aligned}$$

We will now focus on the equation with drift $\lambda X(t)$ and diffusion $\mu X(t)$, where λ and μ are real constants. This will be our test problem:

$$dX(t) = \lambda X(t)dt + \mu dW(t) \ , \ X(0) = X_0 \tag{7}$$

Notice that if $\mu = 0$ there is no *white noise* and simplifies to the Cauchy problem with parameter λ. The value of μ can be interpreted as how random and noisy is the underlying process we want to simulate. Financials use the Black-Scholes model which is derived from this SDE to determine stock prices distributions. In this case, μ would represent the volatility of the stock. Some insights can also be given by resolving it. Using the theorem 1, we can give a heuristic method to give its solution. Heuristic, because we will take $h(x) = ln(x)$ which is not continuous in its first derivative, nor its second. Hence, applying Itô's formula we have

$$\begin{aligned} d\ln(X(t)) &= \frac{1}{X(t)}dX(t) + \frac{1}{2}(-\frac{1}{X(t)^2})\mu^2 X(t)^2 dt \\ &= \frac{1}{X(t)}(\lambda X(t)dt + \mu X(t)dW(t)) - \frac{1}{2}\mu^2 dt \\ &= \lambda dt + \mu dW(t) - \frac{1}{2}\mu^2 dt. \end{aligned}$$

Finally, after integrating and taking the exponential on both sides of the equation, we end up with

$$X(t) = X_0 \exp\left\{(\lambda - \frac{1}{2}\mu^2)t + \mu W(t)\right\}. \tag{8}$$

We can verify that it does correspond to a solution of (7). The following theorem will help us prove that it is in addition, the unique solution.

Theorem 2 (Existence and Uniqueness [1]). *Let $X(t)$ satisfy $dX(t) = \lambda(X(t), t)dt + \mu(X(t), t)dW(t)$. If the following conditions are satisfied*

1. *Coefficients are locally Lipschitz in x uniformly in t, that is, for every T and N, there is a constant K depending only on T and N such that for all $|x|,|y| \leq N$ and all $0 \leq t \leq T$*

$$|\lambda(x,t) - \lambda(y,t)| + |\mu(x,t)\mu(y,t)| < K|x-y|.$$

2. *Coefficients satisfy the linear growth condition*

$$|\lambda(x,t)| + |\mu(x,t)| \leq K(1 + |x|).$$

3. *$X(0)$ is independent of $(B(t), 0 \leq t \leq T)$, and $\mathbb{E}[X(0)^2] < \infty$.*

Then there exists a unique strong solution $X(t)$ of the SDE stated above. $X(t)$ has continous paths, moreover

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} X(t)^2\right) < C(1 + \mathbb{E}(X(0)^2)),$$

where constant C depends only on K and T.

Constant coefficients fulfil trivially the first two conditions. The third one impose requirements on the initial value $X(0)$, therefore we will assume it is sufficiently regular. Therefore, by Theorem 2 (8) is the unique (and strong) solution of (7).

3.3 Approaching an SDE with the θ-method

To apply our numerical method over $[0, T]$, we first need to discretize the interval. For $L \in \mathbb{N}_*$, we write $\delta t = T/L$ and $\tau_j = j\delta t$ with δt representing the size of the steps. The Brownian Motion need therefore to be considered within a discrete time interval. Using all three conditions of the definition we can write $W(\tau_j) = W(\tau_{j-1}) + dW(\tau_j)$ and $W(0) = 0$ where each $dW(\tau_j)$ is distributed as a Gaussian with mean zero and variance δt, i.e $dW(\tau_j) \sim \mathcal{N}(0, \delta t)$. Using the same notations (i.e $f(X_j, \tau_j) = f(\tau_j)$), the θ-method takes the form:

$$X_j = X_{j-1} + \delta t((1-\theta)f(\tau_{j-1}) + \theta f(\tau_j)) + g(\tau_{j-1})(W(\tau_j) - W(\tau_{j-1})), \quad (9)$$

for $j = 1, 2, \cdots, L$ and $X(0) = X_0$. All we did was adding a discretized stochastic term to the method 2.

3.4 Strong and Weak convergence

There exists two ways of measuring accuracy for numerical SDEs: strong and weak convergence. The first one measures the rate at which the mean of the error decreases when $\delta t \to 0$. The second one measures the rate at which the error of the means decrease. It is a less demanding criterion because we only rely upon the expectation of the numerical approximation and not on the path itself.

Definition ([3]). *Let X_n denote the value of the numerical method at the node n. A method is said to have strong order of convergence equal to γ if there exists a constant C such that*

$$\mathbb{E}|X_n - X(\tau)| \leq C\delta t^\gamma$$

for any fixed $\tau = n\delta t \in [0, T]$ and δt sufficiently small.

It will be said to have weak order of convergence equal to γ if there exists a constant C such that for all functions p in some class

$$|\mathbb{E}p(X_n) - \mathbb{E}p(X(\tau))| \leq C\delta t^\gamma$$

at any fixed $\tau = n\delta t \in [0, T]$ and δt sufficiently small. Functions p allowed in this class must satisfy smoothness and polynomial conditions.

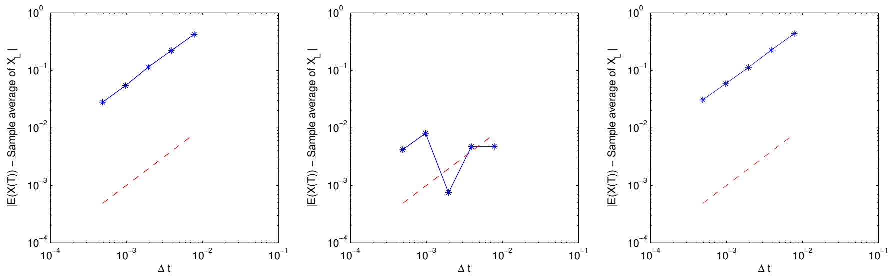

Figure 3: Weak error plot for $\theta \in \{0, 1/2, 1\}$ with a (red) reference slope of 1.

We will concentrate on the case where p is the identity function.

Figure 3 show results of an empirical way to find weak convergence order of the θ-method. These graphs have been done running a matlab program which returns $|\mathbb{E}(X(t)) -$ Sample average of $X|$ versus δt. As it appears on the axis, the lines are on a loglog scale. The five dots connected by the blue line are computed using values in the set $\{2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}\}$ for δt. The program also uses 50000 paths to obtain a rather good expectation (See [3] for more information). Plus, we took $\lambda = 2$ and $\mu = 0.1$.

The slope of the blue line corresponds to the order of convergence because we see the direct consequence of decreasing the step size on the error of the means. That means, when the step size is reduced we are able to determine, by comparing with the red line slope, with which scale the error decreases.

The diffusion parameter had to be small enough, because if we put too much randomness, we obtain results difficult to work with. Again, this procedure is empirical and has its limits. It happened that after changing drift, diffusion or step size (δt), dots did not form a complete straight line and behaved differently. First and last graphs give good insights of a potential order of weak convergence, they tend to follow the same slope as the red line. But what about the strange pattern appearing on the middle graph ? Typically, we are confronted to the limits of numerical investigations about theoretical results. Without considering the latter, by comparing with the red line of slope one, we can empirically assume that the weak order of convergence is 1.

Similarly, Figure 4 shows results for strong convergence order. The program returns Sample average of $|X(t) - X_L|$ versus δt. Results are smoother than before when $\theta = 1/2$ and we can empirically deduce that the θ-method has strong order convergence one half by comparing with the red line of slope one half.

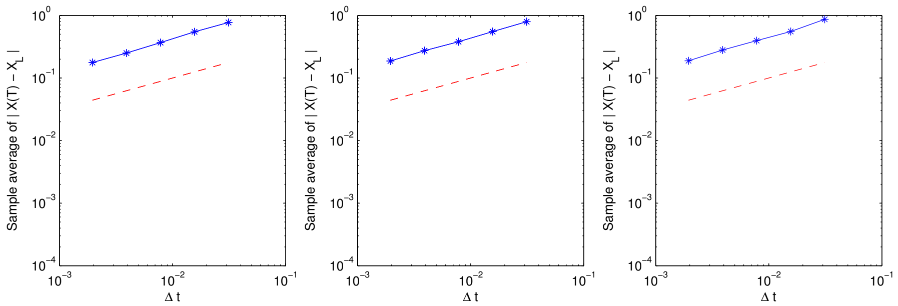

Figure 4: Strong error plot for $\theta \in \{0, 1/2, 1\}$ with a red reference slope of 1/2.

3.5 Mean-square stability

We aim at studying stability of the stochastic theta-method. Because random variables are infinite dimensional object, norms are not equivalent. Therefore, we will apply the mean-square method to measure stability, as it is one of the most common.

Definition. *Assume $X \equiv 0$ is an equilibrium (fixed point, steady state). The trivial solution $X \equiv 0$ of (4) is called globally (asymptotically) p-th mean stable iff $\forall X_0 : \mathbb{E}\|X(0)\|^p < +\infty \implies \lim_{t\to+\infty} \mathbb{E}\|X(t)\|^p = 0$. In the case $p = 2$ we speak of global (asymptotic) mean square stability. Moreover, the trivial solution $X \equiv 0$ of SDE (4) is said to be locally (asymptotically) p-th mean stable iff $\forall \epsilon > 0 \exists \delta : \forall X_0 : \mathbb{E}\|X(0)\|^p < \delta \implies \forall t > 0 : \mathbb{E}\|X(t)\|^p < \epsilon$. In the case $p = 2$ we speak of local (asymptotic) mean square stability. [2]*

What kind of stability domain can we expect from the test problem ? Using Itô's formula with $f(x) = x^2$ (which is $\mathcal{C}^2$), we have

$$dX(t)^2 = (2X(t)^2\lambda + \mu^2 X(t)^2)dt + 2X(t)^2\mu^2 dW(t).$$

Then taking the expectation and recalling that a Brownian Motion has a mean of zero,

$$d\mathbb{E}\left[X(t)^2\right] = \left(2\lambda\mathbb{E}\left[X(t)^2\right] + \mu^2\mathbb{E}\left[X(t)^2\right]\right)dt + 0$$
$$\frac{d}{dt}\mathbb{E}\left[X(t)^2\right] = \mathbb{E}\left[X(t)^2\right]\left(2\lambda + \mu^2\right)$$
$$\mathbb{E}\left[X(t)^2\right] = C\exp^{\left(2\lambda+\mu^2\right)t}.$$

Hence,

$$\lim_{t\to\infty} \mathbb{E}\left[X(t)^2\right] = 0 \iff \Re(2\lambda + \mu^2) < 0 \iff \Re(\lambda) + \frac{1}{2}|\mu|^2 < 0.$$

Which corresponds to

$$S_{exact} = \{\lambda, \mu \in \mathbb{C} : \Re(\lambda) + \frac{1}{2}|\mu|^2 < 0\}.$$

Now that we have the theoretical result, we investigate our numerical scheme. We have to study the behaviour of $\lim_{j\to\infty} \mathbb{E}[X_j]$. For this purpose, we write (9) using $(W(\tau_j) - W(\tau_{j-1})) = \sqrt{\delta t} V_j$ with $V_j \sim \mathcal{N}(0,1)$, since our Brownian Motion follows a Gaussian distribution with variance δt.

$$X_{j+1} = X_j + \delta t((1-\theta)\lambda X_j + \theta\lambda X_{j+1}) + \sqrt{\delta t}\mu V_j X_j$$

$$X_{j+1} = \left(\frac{1 + (1-\theta)\delta t\lambda + \sqrt{\delta t}\mu V_j}{1 - \theta\delta t\lambda}\right) X_j$$

and using notations $p := \delta t\lambda$, $q := |\mu|\sqrt{\delta t}$ the expression reduces to

$$X_{j+1} = \left(\frac{1 + (1-\theta)p}{1-\theta p} + \frac{q}{1-\theta p}V_j\right) X_j.$$

Hence, recalling that $\mathbb{E}[V_j] = 0$ and $\mathbb{E}[V_j^2] = 1$,

$$\mathbb{E}\left[|X_{j+1}|^2\right] = \left(\left|\frac{1+(1-\theta)p}{1-\theta p}\right|^2 + \left|\frac{q}{1-\theta p}\right|^2\right) \mathbb{E}\left[|X_j|^2\right].$$

By recurrence we have

$$\mathbb{E}\left[|X_j|^2\right] = \left(\left|\frac{1+(1-\theta)p}{1-\theta p}\right|^2 + \left|\frac{q}{1-\theta p}\right|^2\right)^j \mathbb{E}\left[|X_0|^2\right]$$

and finally we can find our conditions for mean-square stability :

$$\lim_{j\to\infty} \mathbb{E}\left[|X_j|^2\right] = 0 \iff \frac{|1+(1-\theta)\delta t\lambda|^2 + \delta t|\mu|^2}{|1-\theta\delta t\lambda|^2} < 1.$$

Let us write this equivalence in another way. Again, by taking p and q^2, the expression on the right hand-side becomes $\dfrac{(1+(1-\theta)p)^2 + q^2}{(1-\theta p)^2} \leq 1$. We need only some elementary calculus to write it in an equivalent manner: $q \leq \sqrt{(2\theta - 1)p^2 - 2p}$. Therefore, we will confront parabolical domains (see Figure 5) of stabilities (except for $\theta = 1/2$!). Again, we

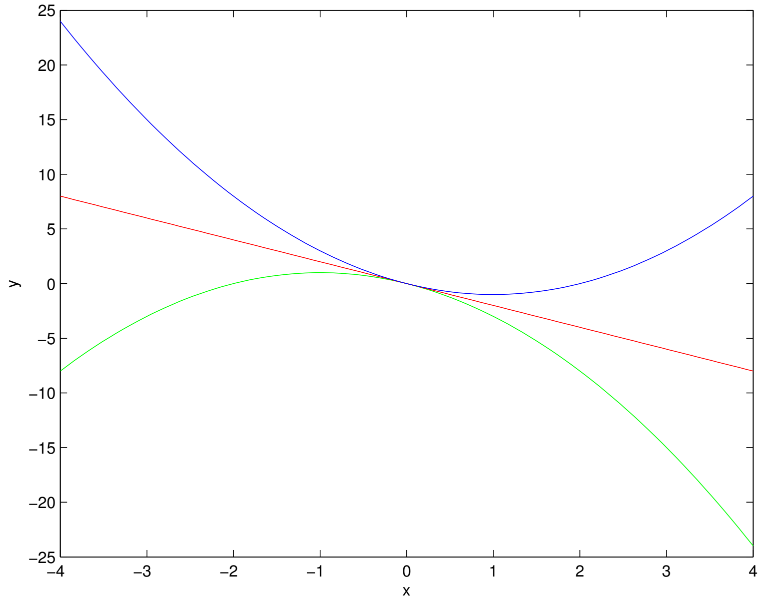

Figure 5: Mean-square stability domain for our three values of theta

4 SDEs with jumps

4.1 Poisson Process

Before giving a definition of a Poisson process, we recall the Poisson distribution and its properties.

Let λ be a strictly positive parameter. A random variable X is said to follow a Poisson distribution if for all $k = 0, 1, 2, \cdots$

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}.$$

X is therefore a discrete random variable and we can show that it has expectation and variance equal to λ. This distribution describes the probability at which a given number of occurrences will take place in a fixed interval of time, assuming occurrences are independent and of known average rate. We can also add its moment generating function: $\mathbb{E}[e^{uX}] = e^{\lambda(e^u - 1)}$.

Definition ([1]). *A Poisson process $N(t)$ over $[0, T]$ is a stochastic process with the following properties.*

1. *(Independence of increments) For $0 \leq s < t < u < v \leq T$ the increments $N(t) - N(s)$ and $N(v) - N(u)$ are independent.*

2. *(Poisson increments) $N(t) - N(s), t > s$, has a Poisson distribution with parameter $\lambda(t-s)$. If $N(0) = 0$, then $N(t)$ has the $Pn(\lambda t)$ distribution.*

3. *(Step function paths) The paths $N(t), t \geq 0$, are increasing functions of t changing only by jumps of size 1.*

Thus we add a variable to Poisson related problems, we count numbers of events but also the time at which they occur. Therefore notions of *inter-arrival* and *waiting time* will be of interest. The first one is the time between consecutive events, the second one for say the k^{th} event, is the time T_k we wait to experience it. We can show that inter-arrivals times are exponentially distributed with parameter λ. For the first event we have

$$\mathbb{P}(T_1 > t) = \mathbb{P}(N(t) = 0) = \mathbb{P}(N(t) - N(0) = 0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t} \tag{10}$$

and similarly, $T_{k+1} - T_k$ has the same distribution [5].

We can note the analogies with the Brownian motion; same kind of properties, same idea but we replaced the normal distribution with the one of Poisson. These two probability laws share some common attributes but are of two different kinds: continuous versus discrete. Brownian motion paths are continuous, it can be seen for example on the approximation from Figure 2. But Poisson process paths experience little *jumps*, they are not continuous, but *càdlàg* which means continuous on the right, limit on the left. This will be reflected in the path geometry of Poisson processes.

In some situations, the last property is too restrictive and inadequate. However, a more broad definition of the poisson process exists and is called a *compound poisson process*. Such a generalisation is made by randomizing the incremental stepsize. Instead of having a fixed increment of one unit, we let the distance follow a distribution f.

Definition ([6]). *A compound Poisson process with intensity $\lambda > 0$ and jump size distribution f is a stochastic process X_t defined as*

$$X_t = \sum_{i=1}^{N_t} Y_i,$$

where jumps sizes Y_i are i.i.d. with distribution f and Nt is a Poisson process with intensity λ, independent from $(Y_i)i \geq 1$.

We can deduce from this definition that

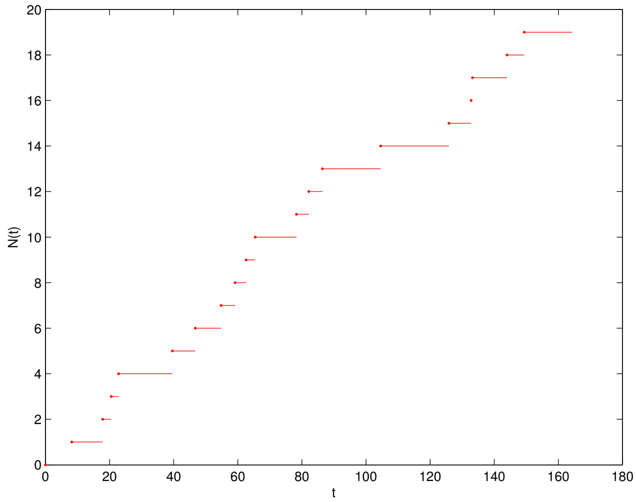

Figure 6: poisson process path with parameter $\lambda = 4$ and fixed to 20 occurrences

4.2 The Merton model

Using the latter concept, we define the Merton model by adding a term to our previous test stochastic equation:

$$dX(t) = \lambda X(t)dt + \mu dW(t) + \sigma \ , \ X(0) = X_0. \tag{11}$$

Such a model is used in finance to determine stocks and bonds prices. Why do we need to introduce jumps ? Well, imagine a quoted company which is about to announce its results of the year. One may expect a big shift on the stock price right after the announcement. Such an event will be called a jump as it is a good visual representation.

References

[1] Fima C. KLEBANER, Introduction to Stochastic Calculus With Applications, Imperial College Press, Melbourne, 2005.

[2] M. ZILI, D. V. FILATOVA, Stochastic Differential Equations and Processes, Springer, Tunisia, 2010.

[3] D. J. HIGHAM, An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations, SIAM Review, Vol. 43, No. 3 (Sep., 2001), pp. 525-546.

[4] Kiyosi ITO, Stochastic processes: lectures given at Aarhus University, Springer, Berlin, 2004, pp 39-50.

[5] Wikipedia contributors, Poisson process, Wikipedia, The Free Encyclopedia, last revision: 8 November 2012.

[6] Rama CONT, Peter TANKOV, Financial Modelling with Jump Processes, CRC press, 2004.