Financial Engineering Project

Numerical Methods for Pricing European and American Options in the Black-Scholes Model with Discrete Dividends

Authors:	*Matr. number:*
Inès Azaiez	08-807-525
Carlos O. Chida Suárez	13-739-131
Svea Ludwig	13-739-065
Yoann Trellu	10-808-657

Supervisors:

Chris Bardgett & Nikola Vasiljević

December 2, 2013

ETH Eidgenössische Technische Hochschule Zürich Swiss Federal Institute of Technology Zurich

University of Zurich[UZH]

Contents

1 Introduction

In this paper, we will investigate methods to price European and American options with discrete dividends using three different numerical methods, the binomial tree approach, the finite difference method (FDM) and the Monte Carlo approach.

First consider the case of dealing with a large portfolio of stocks. In this case one normally introduces a dividend yield instead of discrete dividends to the usual Black Scholes model in order to model option prices. With a single stock however, this approach ceases to work as dividends can no longer be assumed to be continuous (Haug et al., 2003). Consequently, we will not find a closed form solution for option prices for the discrete dividend case.

Instead, using the binomial tree approach and FDM we will implement an idea found in Vellekoop and Nieuwenhuis (2006). Furthermore we will also modify the Monte Carlo method to include those dividends.
In the beginning of each section, we will give a short introduction to each model, followed by an explanation of the implementation. Having implemented the methods, we are then able to discuss advantages and disadvantages of each model, also taking numerical efficiency into consideration.

As a last step, we will compare the three models to each other and argue why we find the binomial tree approach to be most adequate in order to price American and European options with discrete dividends in simplified examples and while the other two methods should be preferred in real life.

2 Importance of Dividends in Option Pricing

In this section, we will discuss some general results of dividends in option pricing as described by Wilmott (2007).

First of all note that in financial theory stock prices are usually assumed to follow a lognormal distribution and that individual stock price paths are modelled as continuous random walks. Once we introduce discrete dividends however, these paths will no longer satisfy the continuity condition since for each dividend payment there will be two discontinuous stock price jumps: one occurring at the announcement date and another when the stock goes ex-dividend.

On the announcement date, we may observe two different possible scenarios: An announced rise in dividends can be seen as good earnings prospects leading to a rise in stock prices. Declining dividends on the other hand could be taken as a sign of financial distress and could consequently result in falling stock prices. This is not always the case however and the dividend irrelevance proposition introduced by (Miller and Modigliani, 1961) even states that dividend policy should not affect stock returns at all. Consequently, we will not consider announcement date jumps in our model and will instead focus on price jumps on the ex-dividend date.

Due to the fact that stock values are considered to be the discounted stream of future dividends, on the ex-dividend one would assume a stock price drop equal to the amount of dividends paid. Clearly, this holds only in an efficient and frictionless market. Note that due to the no-arbitrage condition, the corresponding option value will remain the same.

Another effect which will not be captured in our model is the scenario where the dividend payout date differs from the ex-dividend date. This is for example the case in the UK, where dividends are usually paid out five weeks after the ex-dividend date. In this case, the jump should be equal to the discounted dividend payment. In our model, we will assume that ex-dividend and payout day coincide as is usual for instance in Germany.

3 Pricing of European and American Options in the Black Scholes model

Imagine we want to price a European option V with underlying stock price S, strike price K and maturity T. Assume that the interest rate is given by r and the volatility of the underlying is denoted by σ. In order to work with the Black Scholes Model, we have to make the following assumptions about the market:

- the underlying stock price S follows a log-normal distribution
- the price increments $S_{t+\Delta t} - S_t$ are independent of the past $\mathcal{F}_t$
- the risk-free interest rates r are constant
- the volatility σ is constant
- There are no market frictions
- There are no arbitrage opportunities
- For now: there are no dividends

If these assumptions hold, the standard Black Scholes PDE is then given by

$$\frac{\partial V_t}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_t}{\partial S_t^2} + rS_t \frac{\partial V_t}{\partial S_t} - rV_t = 0$$

Note that this equation holds for call options C whose payoff at maturity is given by $C(S_T, T) = max(0, S_T - K)$ as well as for put options P with final payoff $P(S_T, T) = max(0, K - S_T)$.

For European options without dividends, a closed form solution to the PDE given above can be found. With American options, the situation is a bit more difficult since in this case exercise is possible at every point in time up until maturity. But this means, at each time step it has to be decided if the option value is higher than the payoff one would obtain by exercising the option immediately. Instead of one PDE, as in the case of the European option, we obtain a set of inequalities. For instance, it can be shown that the value of an American put option satisfies the following inequalities:

$$\frac{\partial P}{\partial t}(t, S) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2}(t, S) + rS\frac{\partial P}{\partial S} - rP(t, S) \leq 0$$

$$P(t, S) \geq I^P(S)$$

$$(P(t, S) - I^P(S)) \cdot (\frac{\partial P}{\partial t}(t, S) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2}(t, S) + rS_t\frac{\partial P}{\partial S}(t, S) - rP(t, S)) = 0$$

$$P(T, S) = I^P(S)$$

Now that we have introduced the general Black Scholes model for European and American options and have furthermore talked about discrete dividends in option pricing, let us consider a first approach to implementing prices of options with discrete dividends: the binomial tree approach.

4 The Binomial Tree Approach

4.1 General Idea

As a first step, let us consider the binomial tree approach for European options with one discrete dividend as discussed by Vellekoop and Nieuwenhuis (2006).

The general idea behind this method is to use the usual Binomial Tree approach as discussed by Gilli and Schumann (2009) and to introduce discrete dividends. For European and American options, this is done by making use of the fact that the value of such an option just before a dividend date is equal to the option value for the same stock price minus the dividend amount just after the dividend payment.

4.2 Choice of Parameters and Pricing Formulae

4.2.1 The parameters

In this project we used the Cox Ross and Rubinstein (CRR) approach, in order to choose the parameters of the binomial tree. This approach can be regarded as a discretized version of the Black and Scholes model for pricing European options and relies on matching the mean and variance of the stock's returns in the tree to the mean and the variance of the returns of the lognormal distribution in a continuous time world. A further restriction is introduced in this model and consists in choosing $u * d = 1$. Hence the parameters' setting in the CRR model are

$$\begin{cases} u & = \exp\left(\sigma \cdot \sqrt{\frac{T}{N}}\right) \\ d & = \frac{1}{u} \\ R_f & = \exp\left(r \cdot \frac{T}{N}\right) \\ q & = \frac{R_f - d}{u - d} \end{cases}$$

where u and d are respectively the stock's gross return in case of an upstick and downstick, T is time to maturity, σ is the volatility of the stock, r yearly continuously compounded risk-free interest rate, N number of steps and q is the risk neutral probability of an upstick.

The derivation of these parameters can be found in details in Gilli and Schumann (2009).

4.2.2 Pricing European options

In the CRR settings, we can price European options in a backward fashion using the formula

$$\begin{cases} P(N, S_N) & = \text{ Pay-off at maturity} \\ P(n, S_n) & = \left[q \cdot P(n+1, S_n \cdot u) + (1-q) \cdot P(n+1, S_n \cdot d)\right] / R_f \end{cases}$$

We follow this backward procedure till we obtain the price of the option at time zero, $P(0, S_0)$.

4.2.3 Pricing Amercian options

Amercian options introduce an early exercise feature which is easily implemented in the binomial method. As in the case of European options, pricing relies on a backward formula. However what is requird in the American case, is that when we compute the new option price at a specific node, we need to check whether the pay-off from exercising is greater than the current value of the option. The new option price at each node would then be the maximum of the pay-off and the value of the option at that specific node. The pricing formula for American options is then given by

$$\begin{cases} P_{am}(N, S_N) & = \text{Pay-off at maturity} \\ P_{am}(n, S_n) & = \max\left(\text{ Pay-off at t=n, } \frac{f(n+1, S_n)}{R_f}\right) \end{cases}$$

where the function f is equal to $f(n+1, S_n) = q \cdot P_{am}(n+1, S_n \cdot u) + (1-q) \cdot P_{am}(n+1, S_n \cdot d)$. The following graph clearly illustrates the backward scheme for pricing an amerian put

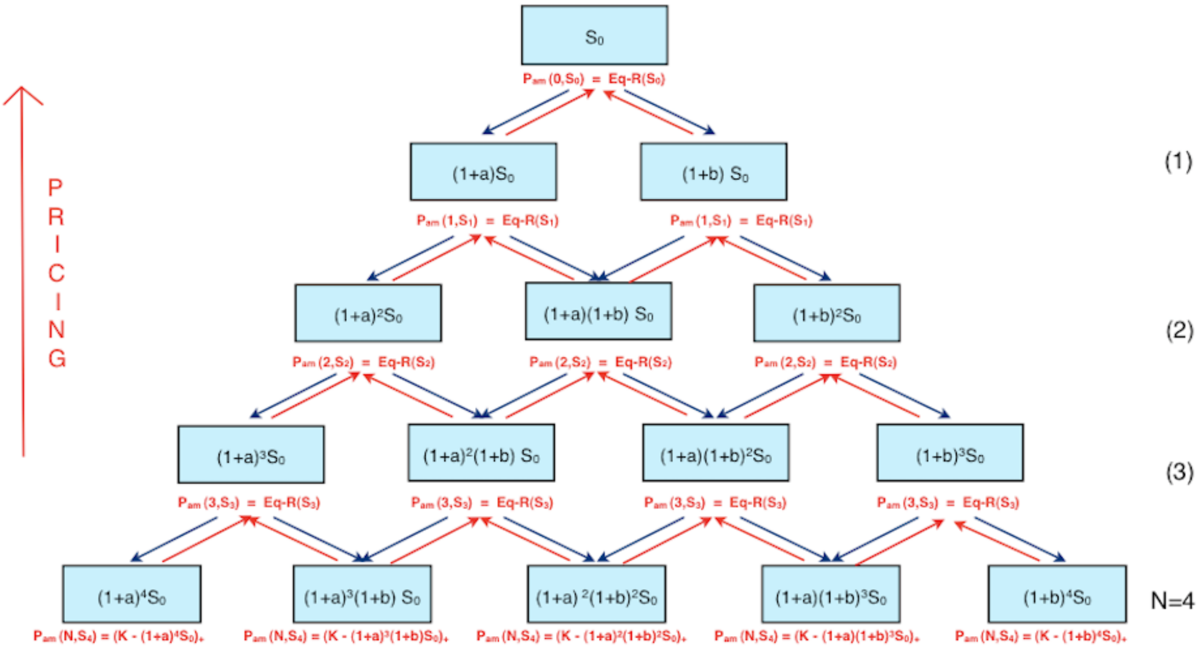

Figure 1: Backward Pricing Scheme for an American put in the Binomial model ($N = 4$)

where $1 + b\ (= u)$ and $1 + a\ (= d)$ denote respectively the uptick and downtick of the stock's price and

$$\text{Eq-R}(S_n) = \max\left((K - S_n)_+, \frac{q \cdot P_{am}(n+1, S_n \cdot u) + (1-q) \cdot P_{am}(n+1, S_n \cdot d)}{R_f}\right)$$

is the pricing formula for the american put.

4.3 Implementation with Dividends

In order to implement the model for European and American options as described by Vellekoop and Nieuwenhuis (2006) consider a binomial tree with n time steps. Denote by s the stock price, a discrete dividend payment at time $m(n)$ by $D(s)$ and the contingent claim by $\Phi(s)$. Since we know the value of $\Phi(s)$ at the end of the tree for all possible stock prices s, we can work backwards as usual to find the set of option values $f^n(s)$ just after a dividend has been paid.

In order to proceed any further however, we need the stock price before the dividend has been paid. This can be derived using the following fact: Since the stock price jumps down by the amount of the dividend at the time of the dividend payment, we know that

$$\begin{gathered}\text{value of option just before dividend payment for underlying stock price } s \\ = \\ \text{value of option after dividend payment for underlying stock price } s - D(s)\end{gathered}$$

Recalling that we work backwards, this means we now have to find $f^n(s - D(s))$ while at the moment we only know $f(s)$. In order to obtain values for $f^n(s - D(s))$, we therefore introduce the interpolating function $B^n_{f^n}$ which gives an approximation for $f^n(s-D(s))$. In particular, we use a cubic spline approach to garantuee a smooth interpolating function. We thus obtain approximating values $B^n_{f^n}(s - D(s))$ from which onwards we can continue our calculations on a new, shifted, binomial tree.

4.4 Results

4.4.1 European Call Option

Implementing the Black Scholes model using the binomial tree approach with discrete dividends as explained above, we obtain results for European Call options as can be seen in Table 1 below. In order to be consistent with the example chosen by Vellekoop and Nieuwenhuis (2006), we assume a stock price of $S = 100$, a maturity of 7 years and 7 discrete dividend payments of value 6.0, 6.5, 7.0, 7.5, 8.0, 8.0, 8.0. Note that the first payment occurs in time t_1 for $t_1 \in \{0.1, 0.5, 0.9\}$ while consecutive dividends are paid out yearly thereafter. We furthermore assume a volatility of $\sigma = 25\%$ and interest rates of $r = 6\%$. For the strike price we consider the cases when K takes values 70, 100 and 130.

t_1	K	250 Steps	500 Steps	1000 Steps	VN 1000	abs. diff.	% diff.
0.10	70	24.931199	24.908775	24.906608	24.92	-0.01	-0.1
0.10	100	17.440670	17.438198	17.436515	17.46	-0.02	-0.1
0.10	130	12.404865	12.409789	12.403916	12.43	-0.03	-0.2
0.50	70	26.111993	26.092030	26.090711	26.10	-0.01	0.0
0.50	100	18.484606	18.484584	18.483919	18.50	-0.02	-0.1
0.50	130	13.286150	13.293525	13.288689	13.31	-0.02	-0.2
0.90	70	27.245566	27.224904	27.223121	27.23	-0.01	0.0
0.90	100	19.486240	19.484804	19.483568	19.50	-0.02	-0.1
0.90	130	14.132781	14.138665	14.133259	14.16	-0.03	-0.2

Table 1: Results for the value of European Call options with different times of dividend payments, strike prices and time steps

As can be seen in Table 1, the replication of the results by Vellekoop and Nieuwenhuis (2006) worked, independently of the time steps or strike prices used, very well. Considering the absolute differences between our model with 1000 time steps ("1000 Steps") and the paper's 1000 time step model ("VN 1000") as can be seen in column 7 of Table 1 ("abs. diff."), we obtain that the results differ never more than 3 dollar cents. In terms of percentage difference (column 8, "% diff.") this means our results never deviate from the paper's result by more than 0.2%.

Taking different number of time steps into consideration, we obtain that the more time steps used in the model, the closer are our result to the paper's. This makes sense as increasing the number of time steps always leads to an increase in accuracy. In our case, we can normally increase the accuracy by up to \$0.01 by going from 250 to 500 steps (e.g. from 12.40 to 12.41 for $t_1 = 0.1$ and $K = 130$).

Another interesting thing to note is that our values are always slightly smaller but never exceed the paper's results. This could be related to the fact that we chose to use the floor function instead of rounding to obtain the equivalent steps corresponding to the payout times. This means, that if a dividend is paid at time 0.75 using 365 steps per year, it would be set to the step number 273 instead of 274 even when the exact calculation is $0.75 \times 365 = 273.75$. This chosen function allows us to keep a better control over the memory allocation.

Another explanation for this small difference between our results and those presented in the paper might be due to the use of another interpolating function. As mentioned previously, in our project we used the spline interpolating function. The authors of the paper did not specify which interpolating function they used.

4.4.2 American Call Option

For the American Call option, we still use a strike price of $S = 100$ but now assume a volatility of $\sigma = 30\%$ and an interest rate r of 5% in order to be consistent with the paper by Vellekoop and Nieuwenhuis (2006). Also note that this time we only consider a single dividend payment of value 7.0, payable again at times 0.1, 0.5 or 0.9. The results of our implementation can be found in Table 2 below.

t_1	K	250 Steps	500 Steps	1000 Steps	VN 1000	Abs. diff.	% diff.
0.10	70	30.366929	30.375520	30.379850	30.38	0.00	0.0
0.10	100	10.278340	10.284356	10.287349	10.29	0.00	0.0
0.10	130	2.988048	2.994817	2.994643	3.00	-0.01	-0.2
0.50	70	32.112626	32.124030	32.127754	32.12	0.01	0.0
0.50	100	11.307582	11.316869	11.321195	11.32	0.00	0.0
0.50	130	3.270816	3.277838	3.278352	3,28	0.00	-0.1
0.90	70	33.897445	33.908592	33.911830	33.91	0.00	0.0
0.90	100	13.468975	13.476266	13.485265	13.48	0.01	0.0
0.90	130	4.152394	4.161930	4.164719	4.16	0.00	0.1

Table 2: Results for the value of American Call options with different times of dividend payments, strike prices and time steps

First of all, note that in some cases the value of the American option is lower than the value of the European option we considered in Section 4.4.1 (e.g. for dividend time $t_1 = 0.1$ and strike price $K = 100$, we as well as the paper obtain a European Call price around 17.44 while for the American call we get a price of around 10.29). This however can be explained by the different dividend payments used in the two scenarios (recall that for the European option, there were 7 dividend payments while now we only have 1).

In terms of accuracy, for the American call we get even closer to the paper's results. While most of the time the deviation amounts to less than a dollar cent, the maximum being \$0.01 away, the percentage difference is consequently never higher than 0.02%.

As before, we have that accuracy increases with the number of time steps used. Again, we observe an increase in value of around 1 dollar cent from 250 to 500 steps. Note that doubling the time steps from 500 to 1000 only increases the value by another \$0.004. This does not sound much at first, however considering that often many options worth several million dollars are bought, one can argue that even small improvements matter.

4.5 Advantages and Disadvantages

Having implemented the binomial tree approach and having furthermore compared the resulting option values with the ones given by Vellekoop and Nieuwenhuis (2006), one can say that this model works very well for our example. It is easy to implement, fast and gives very accurate results. It is very intuitive in comparaison to other numerical pricing methods (FDM) and can easily handel early exercise opportunities.

One thing to note however is the simplicity of the underlying's possible states. At each node of the binomial tree, there are only two possibilities for the stock price: up or down. Clearly, this is not very close to reality. Additionally, it is fairly difficult to implement additional features in the binomial model, such as e.g. jumps.

5 The FDM Approach

5.1 General Idea

The general idea behind the Finite Difference Method (FDM) is to create a two dimensional grid (in the pricing of an option having only one underlying) where nodes represent a potential stock price S at time t. Knowing the option price at maturity, we can work backwards in time to fill in the grid with option prices using the Black-Scholes PDE and finite differences to approximate the derivatives. This method differs from the previous one by the way the dynamics is given; A binomial tree approach will have almost all the dynamics, except for the probability q, included in the construction of the tree, whereas the FDM scheme includes the dynamics only via the PDE.

Note that instead of having linear increments for S in the grid, we could use logarithmic increments. Such a construction is motivated by the fact that stock prices are assumed to follow a lognormal distribution. However the PDE will be slightly changed.

Additionally, Wilmott (2007) points out that many improvements on accuracy and speed have been implemented for the finite difference method while this is not the case for the binomial tree approach.

In particular, we will use the Crank Nicholson form of the FDM as discussed by Wilmott (2007) to find an approximate solution to the Black Scholes formula as described in the following section.

5.2 Implementation

As a first step, let us consider a grid with equal spaces between nodes. In this grid we have equal time intervals on the abscissa of value δt and equal stock price intervals on the ordinate denoted by δS. Note that similarly to the binomial tree approach, we will work backwards and hence will consider time values

$$t = T - k\delta t \text{ for } k = 0, 1, ..., K$$

where T denotes the maturity of the option. This means we start at maturity T and work backwards $(T - 1\delta t, T - 2\delta t, ..., T - k\delta t, ...)$ until we arrive at time t. The stock price S, given by

$$S = i\delta S \text{ for } i = 0, 1, ..., I$$

can in theory be arbitrarily high meaning that I could be possibly infinite. In practice however, we will use three to four times the stock price at time t_0.

The value of the option for an underlying stock price at node i at time step k can then be described by

$$V_i^k = V(i\delta S, T - k\delta t)$$

To understand how we obtain option value V_i^k from option values later in time, consider the Black Scholes formula:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

where S denotes the stock price, V the value of the option, σ is the volatility of the stock and r denotes the risk-free interest rate. Furthermore note that $\frac{\partial V}{\partial t}$ is θ, the partial derivative of V with respect to the time t and $\frac{\partial V}{\partial S}$ as well as $\frac{\partial^2 V}{\partial S^2}$ denote the first and second partial derivatives of V with respect to stock price S, in greek notation called Δ and Γ respectively. Note that in the following, we will generalize the PDE to ease the reading:

$$\frac{\partial V}{\partial t} + a(S,t)\frac{\partial^2 V}{\partial S^2} + b(S,t)\frac{\partial V}{\partial S} + c(S,t)V = 0$$

We will approximate the partial derivatives in the Black Scholes formula by applying the finite difference method. In particular, we will use the Crank Nicholson approach which can be seen as a mixture between the explicit and the implicit approach of FDM. While the explicit method uses option values V_{i-1}^k, V_i^k and V_{i+1}^k in order to infer the value of one time step before, i.e. the value of V_i^{k+1}, the implicit method does the opposite: Here one option value V_i^k is used to derive three values for the next step k, i.e. V_{i-1}^{k+1}, V_i^{k+1} and V_{i+1}^{k+1}. Graphically, this is depicted in Figures 2 and 3.

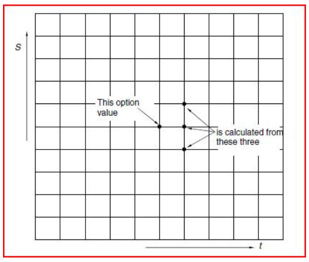

Figure 2: *Relationship between option values under the explicit method*

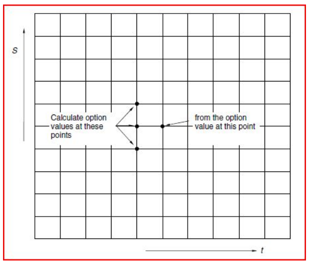

Figure 3: *Relationship between option values under the implicit method*

In the following we will explain how θ, Δ and Γ are computed in the explicit and implicit method and will consequently arrive at the Crank Nicholson approach to solving our problem.

5.2.1 Approximation of θ

For the approximation of θ, explicit and implicit method use the same approach as explained in the following. Since θ is the first derivative of the option value V with respect to time we have that

$$\frac{\partial V}{\partial t} = \lim_{h \to 0} \frac{V(S, t+h) - V(S, t)}{h}$$

This can be approximated by the following formula:

$$\frac{\partial V}{\partial t}(S, t) = \frac{V_i^k - V_i^{k+1}}{\delta t}$$

5.2.2 Approximation of Δ

For the approximation of Δ, one can use forward, backward or central differences. Since according to the Taylor expansion, the central difference method will lead to the highest accuracy, we will use this method. Consequently, we will use the following formula for obtaining Δ in the explicit case:

$$\frac{\partial V}{\partial S}(S, t) = \frac{V_{i+1}^k - V_{i-1}^k}{2\delta S}$$

Recalling that for the explicit method we will use time k while for the implicit method we obtain the same term at time $k + 1$, we get the following approximation for Delta under the implicit method:

$$\frac{\partial V}{\partial S}(S, t) = \frac{V_{i+1}^{k+1} - V_{i-1}^{k+1}}{2\delta S}$$

Note that for each V_i^k we need to know $S + \delta S$ and $S - \delta S$ at $k+1$ which is not possible at the boundaries of the stock price. This does not impose a problem however as one can use the forward or backward difference method at these points instead.

5.2.3 Approximation of Γ

Recall that Γ is the second derivative of the option value with respect to the stock price. It is usually approximated by first computing the difference between the forward and backward differences and then dividing by δS. Mathematically, this becomes

$$\frac{\partial^2 V}{\partial S^2}(S,t) = \frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{\delta S^2}$$

As before, the implicit method only differs in the time index and hence the equation for this method becomes

$$\frac{\partial^2 V}{\partial S^2}(S,t) = \frac{V_{i+1}^{k+1} - 2V_i^{k+1} + V_{i-1}^{k+1}}{\delta S^2}$$

5.2.4 Boundary conditions

Since the finite difference method is using a grid framework, we need bounday conditions for our option values. Due to their different nature, call and put options require different conditions.

Let us first consider call options C. In this case, we know that for a stock price S of 0, the option becomes worthless. Consequently, we have that

$$C_0^k = 0.$$

For very large stock prices S on the other hand, we have that, up to exponentially small terms, the value of a call option converges to $C = S - Ke^{-r(T-t)}$. Therefore we will use the following condition for the upper bound in our analysis:

$$C_I^k = I\delta S - Ke^{-rk\delta t}.$$

For the put option, the situation is reversed. Now we have that very high stock prices mean that the option has no value. Hence we obtain the following upper boundary condition for the value P of put options:

$$P_I^k = 0.$$

Additionally, we have that for $S = 0$, the value of the put option is given by $P = Ke^{-r(T-t)}$ which in our notation means that

$$P_0^k = Ke^{-rk\delta t}.$$

5.2.5 The Crank Nicholson Method

Now that we understand how to approximate θ, Δ and Γ using the explicit and the implicit methods, we can also derive the formula for Black Scholes under the Crank Nicholson method. For this, recall that the Crank Nicholson method is just a mixture of the other two methods. Graphically, this is displayed in Figure 4 below:

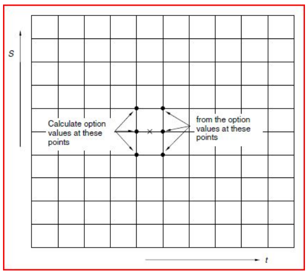

Figure 4: *Relationship between option values using the Crank Nicholson method*

In particular, for each of Δ and Γ one uses a weighted sum of the results of the explicit and the implicit sum with equal weights. Consequently, we obtain the following equation for the option value:

$$\frac{V_i^k - V_i^{k+1}}{\delta t}$$
$$+\frac{1}{2}a_i^{k+1}\frac{V_{i+1}^{k+1} - V_{i-1}^{k+1}}{2\delta S} + \frac{1}{2}a_i^k\frac{V_{i+1}^k - V_{i-1}^k}{2\delta S}$$
$$+\frac{1}{2}b_i^{k+1}\frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{\delta S^2} + \frac{1}{2}b_i^k\frac{V_{i+1}^{k+1} - 2V_i^{k+1} + V_{i-1}^{k+1}}{\delta S^2}$$
$$+\frac{1}{2}c_i^{k+1}V_i^{k+1} + \frac{1}{2}c_i^k V_i^k = O(\delta t^2, \delta S^2)$$

Rearranging this equation in such a way as to obtain option values V^k on the left hand side and all option values V^{k+1} on the right hand side, we obtain the following expression:

$$-A_i^{k+1}V_{i-1}^{k+1} + (1 - B_i^{k+1})V_i^{k+1} - C_i^{k+1}V_{i+1}^{k+1} = A_i^k V_{i-1}^k + (1 + B_i^k)V_i^k + C_i^k V_{i+1}^k \tag{1}$$

for

$$A_i^k = \frac{1}{2}v_1 a_i^k - \frac{1}{4}v_2 b_i^k$$

$$B_i^k = -v_1 a_i^k + \frac{1}{2}\delta t c_i^k$$

$$C_i^k = \frac{1}{2}v_1 a_i^k + \frac{1}{4}v_2 b_i^k$$

Note again that these equations are only valid for $1 \leq i \leq I - 1$ while for the remaining two cases the boundary conditions have to be used.

Having found the linear system (1), we now need to solve it. Let us rewrite this system in a nicer way as follow:

$$M_L v^{k+1} = M_R v^k + R^k. \tag{2}$$

We can drop some of the time indexes, since in the case of the Black Scholes PDE, equation (1) has vectors A, B, C independent of time. Let us also remark that now v^k is the option price for the inner points of the grid at time $T - k\delta t$, and R^k is the term incorporating the boundary conditions along S_{min} and S_{max}. Since we know the right hand side, and M_L is tridiagonal by construction, we are reduced to solve the standard linear system $Ax = b$ with A being nice enough to use for example the *LU* decomposition. In practice, we call the *linsolve* command in Matlab.

The latter method can be used to price European option, but the American ones require a bit more work since we have to incorporate the *Early-Exercise* property in the linear system.

5.2.6 SOR to overcome American option pricing

Direct methods are not suitable for American option pricing within the Crank Nicolson scheme for the following reason: Since it incorporates an implicit part that is, values within the vector v_k depend on other values from the same vector, we cannot explicitly include the condition $v_k = \max(v_k,\ \text{exercising now }).$

As mentioned, the *Early-Exercise* property does not allow us to use direct methods to solve the linear system (2). *Direct* method here would mean a solver which only requires a finite number of steps, as opposed to *iterative* methods which can theoretically require an infinite number of steps. Here is a bit of theory on the latter:

To solve $Ax = b$, we set up a sequence $(x_k)_{k\geq 0}$ of vectors that converges to the exact solution that is,

$$\lim_{k\to\infty} x_k = x$$

for any given starting value x_0. A possible strategy could be to solve

$$x_{k+1} = Bx_k + g\ ,\ k \geq 0 \tag{3}$$

with a vector g making the relation consistent (i.e $x = Bx + g$). Furthermore, the convergence of $(x_k)_{k\geq 0}$ requires assumptions on B, for example when the matrix is symmetric and has a *spectral radius*, $\rho(B)$, strictly smaller than one, we have convergence. Further developments can be found in A. Quarteroni (2010).

Iterative methods provide a solution to the early exercise problem within the Crank Nicolson method. Indeed, with the following slight modification of (3) we are able to include our condition.

$$\begin{aligned} x_{k+1} &= Bx_k + g \\ x_{k+1} &= \max(x_{k+1},\ \text{payoff if exercised })\ ,\ k \geq 0. \end{aligned}$$

In practice we solve our system $Ax = b$ with $A = D + L + U$ its decomposition in diagonal, lower and upper parts as follow:

$$\begin{aligned} x_{k+1} &= x_k - \omega(L + D)^{-1}((L + D + U)x_k - b), \\ x_{k+1} &= \max(x_{k+1},\ \text{payoff if exercised })\ ,\ k \geq 0 \end{aligned}$$

where ω is the *overrelaxation* parameter strictly lying between 0 and 2 to have convergence. We stop updating when the error $e^{k+1} = \|x^{k+1} - x^k\|$ is smaller than some constant tolerance fixed at the beginning. To give an idea of how the scheme behaved to produce the results given below, about 300 iterations were made at each time step to pass a tolerance of 10^{-6}, with $\omega = 1.1$. When we decreased the tolerance to 10^{-12}, the number of iterations went up to more than 700. To be noted, these values changed significantly when the risk free rate and sigma changed. This is understood by the fact that these two parameters affect the conditioning of the matrix B in (3) which depends on A (which in turn depends on r and σ). For further developments, this scheme can be found in (W. H. Press, 1992).

5.2.7 Incorporation of dividends

For the finite difference method, we basically use the same technique as in the Binomial tree method. Doing so is motivated by the fact that the Binomial tree method is on many aspects similar to the FDM. Indeed in the case of the explicit scheme, we use tree nodes forward in time to derive the call price, as would do a multinomial tree of order three. Now to be more precise, we fit a spline on the call prices for the different values of stock prices at the dividend time, and update these call prices with the extrapolation accounting for the dividend payoff via the spline. To be noted, the motivation for using the spline instead of other extrapolation methods is only based on its nice properties on the trade off between accuracy and smoothness, and a negligible computational cost.

Now, the main drawback of this method used in both FDM and Binomial tree, is that we no longer have continuity of the option price (that we should have at least in the European case). Yet, the continuity of option prices is explained with a forward looking approach whereas both methods are backward looking. Indeed, to explain the continuity in the European case, we use an information argument. At time t_0, knowing future dividends behavior (motivating this forward looking approach), the option price will already incorporate this information and should therefore not account for any movements in the underlying at the time of dividend. An investor should therefore have no incentive to buy nor sell the option around the time of dividend.

5.3 Results

5.3.1 European Call Option

Now let us look at some results when using the finite difference method. Analogously to the implementation of the Binomial Tree approach, we first considered a European call option with an underlying current stock price $S_0 = 100$, a maturity of 7 years and 7 discrete dividend payments as before, payable once every year with three different first payment dates t_1. Again, we assumed a volatility of 25%, an interest rates of 6% and three different strike prices K, as can be seen in Table 3 below.

t_1	K	250 Steps	500 Steps	1000 Steps	VN 1000	Abs. diff.	% diff.
0.10	70	24.346234	24.354197	24.358178	24.92	-0.56	-2.3
0.10	100	17.033347	17.040386	17.043905	17.46	-0.42	-2.4
0.10	130	12.122713	12.128702	12.131696	12.43	-0.30	-2.4
0.50	70	25.610330	25.616618	25.619761	26.10	-0.48	-1.8
0.50	100	18.155763	18.161385	18.164196	18.50	-0.34	-1.8
0.50	130	13.079596	13.084443	13.086867	13.31	-0.22	-1.7
0.90	70	26.836411	26.843502	26.847046	27.23	-0.38	-1.4
0.90	100	19.245251	19.251621	19.254806	19.50	-0.25	-1.3
0.90	130	14.011986	14.017511	14.020273	14.16	-0.14	-1.0

Table 3: Results for the value of European Call options with different times of first dividend payments t_1, strike prices K and time steps

Considering our results in Table 3 above, one can argue that the Crank Nicholson FDM approach yields coherent results. As with the binomial tree approach, our results are slightly below the paper's results but the deviation is consistent across different dividend payment times t_1 and different strike prices K. This time however, the difference is bigger than with the binomial tree approach. For instance, for a strike price K of 70 and a first dividend payment at time 0.1, Vellekoop and Nieuwenhuis (2006) computed an option value of 24.92 while the FDM produces a value of only 24.36. This means that in this case our value lies \$0.56 below the paper's value. Note that on average the difference in US Dollar amounts to -0.34 (as compared to -0.02 in the binomial case).

As before, doubling the amount of time steps from 250 to 500 leads to an increase in value of around \$0.01, getting the result closer to the paper's. However, even using 1000 steps we're still on average 1.8% below the paper's values.

5.3.2 American Call Option

For the American call option, we use the same settings as with the binomial tree approach in Section 4.4.2 ($S_0 = 100$, $\sigma = 30\%$, $r = 5\%$, $K \in (70, 100, 130)$, $D = 7.0$ paid at time $t_1 \in (0.1, 0.5, 0.9)$). The results using 250, 500 and 1000 steps in the finite difference method can be found in Table 4 below.

t_1	K	250 Steps	500 Steps	1000 Steps	VN 1000	Abs. diff.	% diff.
0.10	70	28.099337	28.121448	28.132273	30.38	-2.25	-7.4
0.10	100	9.761300	9.763365	9.764398	10.29	-0.53	-5.1
0.10	130	2.783473	2.784799	2.785461	3.00	-0.21	-7.2
0.50	70	30.370085	30.392071	30.403191	32.12	-1.72	-5.3
0.50	100	10.548247	10.561549	10.568232	11.32	-0.75	-6.6
0.50	130	3.040420	3.042386	3.043403	3.28	-0.24	-7.2
0.90	70	32.766141	32.791030	32.803407	33.91	-1.11	-3.3
0.90	100	12.763016	12.789004	12.801543	13.48	-0.68	-5.0
0.90	130	3.856695	3.870829	3.877569	4.16	-0.28	-6.8

Table 4: Results for the value of American call options with different times of first dividend payments t_1, strike prices K and time steps

Considering the results in Table 4, we can see that for the American call option, the FDM produces less accurate results than for the European call option. While the results in the European case were already a lot worse than with the binomial tree method, for the American they are significantly below the paper's values.

For a strike price K of 70 and a dividend payment occurring at time 0.10, our result deviates from the paper's result by \$2.25 which in terms of percentages amounts to 7.4%! Unfortunately, this is not the only deviation; on average the absolute difference amounts to $-\$0.86$ or -6.0%.

5.4 Advantages and Disadvantages

First of all note that the finite difference method has the advantage of being general enough to be used in many engineering sectors. Its popularity is mainly due to its simplicity of implementing and understanding. Especially for small dimensions, such as time and one stock price, the FDM yields adequate results in practice. For higher dimensions however, the Monte Carlo method which is explained in Section 6 is more appropriate. Another advantage of the FDM is its ability to handle embedded decisions such as early exercise rights.

Yet, we are quite disappointed by the relatively poor results of this method when discrete dividends are included. We still have coherent results but the systematic error could not be solved with an increase in the precision. This fact is not encouraging and is against the hypothesis that the implementation idea for dividends provided in Vellekoop and Nieuwenhuis (2006), can also be implemented in an FDM approach.

As a continuation of this project, a good target would have been to prove that this idea is also valid (or invalid) within the explicit finite difference method. For doing so, one could start by proving it can also work in a multinomial tree of order three and then try to imply it for the explicit FDM.

Another slight disadvantage compared to the binomial tree, is its computational cost. Pricing American options using the SOR method required significantly more time due to the iterative method to solve for each time step.

5.5 Numerical Efficiency

Since the grid we fill is in two dimensions, the convergence rate will depend on both the step size for the time and for the stock price. In landau notation we have order $O(\delta t^2, \delta S^2)$ with the Crank-Nicolson scheme.

6 The Monte Carlo Approach

Throughout the following lines, we will explain our reasoning using the Monte Carlo approach to deal with the problem of computing a *fair* price for an European option. The case dealing with American options is not stated due to the different nature (forward looking) of Monte Carlo simulations and therefore the need of an extended option-pricing theory which is not mastered by us. Nevertheless, we can tell that there is not one single approach and can vary among computational, financial, mathematical and economical theories and it is to the date a research topic.

If some of the theoretical framework is overlooked, it is due to pursuing a better understanding of the algorthim instead of a full study of the probabilistic theory behind it. These gaps can be filled by refering to Glasserman (2004).

6.1 General Idea

Monte Carlo methods are based on the analogy between probability and volume, this means, the idea that the probability of an event is the number of positive outcomes over the number of feasible ones. For instance, consider the quantity α as follows

$$\alpha = \int_0^1 f(x)dx.$$

The last integral can be also seen as $\mathbb{E}[f(U)]$ where $U \sim \text{Unif}(0,1)$. Therefore, if we could obtain infinitely many realizations of U, the convergence $\hat{\alpha}_n \to \alpha$ as $n \to \infty$ for

$$\hat{\alpha}_n = \frac{1}{n}\sum_{i=1}^{n} f(U_i).$$

is guaranteed by the central limit theorem. In our specific context, the utility of this result is applied further.

The Black-Scholes models states that an underlying assets follows the following stochastic partial differential equation

$$dS(t) = S(t)\left[r\,dt + \sigma\,dW(t)\right],$$

where W(t) is a standard Brownian motion, r the drift of this Itô's process and σ its volatility being both constants. If we assume that the drift r, which represents the rate of return of the underlying, is the same as the risk-free rate, it yields

$$S(T) = S(0)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma W(T)\right]$$

for the security price at a maturity time $T > 0$ as the dynamics under a *risk-neutral* probability measure. Moreover, since $W(T) \sim \text{N}(0,T)$ we can rewrite the previous equation as

$$S(T) = S(0)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z\right],$$

where Z is a standard normal distributed random variable.

Recalling that the value of a European call option for instance, $(S_T - K)_+ := \max(S_T - K, 0)$, we say that its value $C := \mathbb{E}_{\mathbb{Q}}\left[e^{-rT}(S_T - K)_+\right]$ can be approximated by

$$\hat{C}_n = \frac{1}{n}\sum_{i=1}^{n} e^{-rT}\left(S(0)\exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}z_i\right] - K\right)_+,$$

where z_i are independent realizations of a standard normal random variable.

Nevertheless, our problem turns out to be more complicated than just applying the equation above since our underlying asset does not follow the log-normal dynamics assumed in the Black-Scholes models but piecewise due to the dividend payouts. The solution proposed to solve this issue is better explained by implementing an algorithm.

6.2 Implementation

For the sake of visual appreciation, we implemented an algorithm to create complete sample paths of a geometric Browninan motion. This task is summarized in the following steps:

1. Create an $m \times n$ matrix $\tilde{S}$ whose entries are independent realizations of a standard Brownian motion. In this case the number of rows m is the number of steps we want to observe and the number of columns n is the number of realizations.
2. Compute $\exp\left\{\left(r - \frac{\sigma^2}{2}\right) dt + \sigma\sqrt{dt}\tilde{S}_{i,j}\right\}$ for all the matrix entries $\tilde{S}_{i,j}$, where dt denotes the size of the time step.
3. Create an $(m+1) \times n$ matrix S such that $S_{1,j} = 1$ for all $j = 1, \ldots, n$ and $S_{i,j} = \prod_{k=1}^{i} S_{k,j}$ for all $i = 2, \ldots, m$, this means the commulative product along columns.
4. Multiply each entry by $S(0)$.

Now that we know how to generate a matrix corresponding to sample paths of a geometric Brownian motion, we will do it for the number of steps corresponding with the first dividend payout. Then, we *apply* the dividends using a liquidator convention which means that if the stock price is less than the dividend, the company delivers its whole remaining value to he shareholders, in other words $S(t_D^+) = (S(t_D^-) - D)_+$ where t_D^+ and t_D^- represent the moments right before and right after the dividend payout.

Finally, we repeat the algorithm stated at the begining of this subsection but this time we simulate just the continuation of each path and multiply by $S(t_D^+)$ and concatenate this results into a new matrix. This new matrix is now concatenated with the previous one making sure that the row corresponding to t_D^+ replaces the one corresponding to t_D^-. We continue doing this until the maturity time T. Note that we work under the assumption of no change on the volatility or the risk-free rate for the *ex dividend* times.

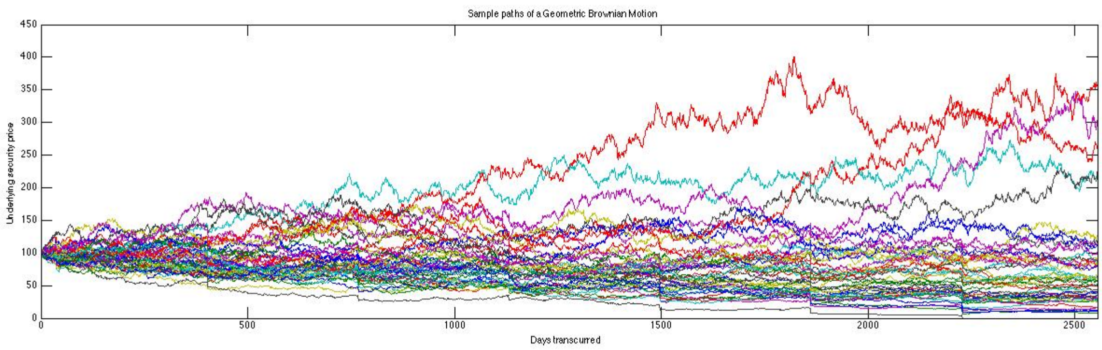

Figure 5: First 50 sample paths of a geometric Browninan motion with dividends payout at times 0.1, 1.1, . . . , 6.1.

Now that we have generated the sample paths with the dividend payouts, we obtain the option prices for each of the paths and get their mean. However, the number of simulations may not be enough the first time to state some kind of convergence. Thus we need to accomplish both, generate enough simulations to assure a good estimation and to guarantee the finiteness of the algorithm. This is achieved by the following reasoning; we will generate paths until the estimator $\hat{C}_n$ fulfills some desirable statistical properties or the computer memory is full. The first one is observable by checking the software documentation and is completely deterministic. On the other hand, the second is achieved by the following calculations; let μ and σ be the mean and standard deviation of C_n, we want that for α and ε positive and relatively small the following holds,

$$\mathbb{P}\left((1-\varepsilon)\mu < \hat{C}_n \leq (1+\varepsilon)\mu\right) \geq 1 - \alpha$$

which yields

$$\mathbb{P}\left(-\frac{\varepsilon\mu}{\sigma/\sqrt{n}} < \frac{\hat{C}_n - \mu}{\sigma/\sqrt{n}} \leq \frac{\varepsilon\mu}{\sigma/\sqrt{n}}\right) \geq 1-\alpha$$

and this is accomplished when

$$\Phi\left(1-\frac{\alpha}{2}\right) \leq \frac{\varepsilon\mu}{\sigma/\sqrt{n}}$$

or equivalently

$$n \geq \frac{\sigma^2}{\varepsilon\mu^2}\Phi\left(1-\frac{\alpha}{2}\right).$$

Therefore the algorithm must stop when the following condition is fulfilled

$$n \geq \min\left(\frac{\hat{\sigma}^2}{\varepsilon\hat{\mu}^2}\Phi\left(1-\frac{\alpha}{2}\right), n_{\max}\right),$$

where $\hat{\mu}$ and $\hat{\sigma}^2$ are the sample mean and standard deviation and $n_{\max}$ is the deterministic maximum number of simulations possible due to hardware constraints to assure either some kind of stochastic convergence or the finiteness of the algorithm in the worst case.

6.3 Results

6.3.1 European Call option

Let us now investigate how well the Monte Carlo method does in our European Call example. In order to be able to compare the results to the paper by Vellekoop and Nieuwenhuis (2006) and also to the binomial tree approach and the finite difference method implemented in Sections 4.4.1 and 5.3.1, we will use the same parameters as before.

The results from the Monte Carlo approach can be found in Table 5 below.

t_1	K	# Simulations	Results	VN 1000	Abs. diff.	% diff.
0.10	70	267,264	24.8859	24.92	-0.03	-0.1
0.10	100	440,320	17.3832	17.46	-0.08	-0.4
0.10	130	673,792	12.3633	12.43	-0.07	-0.5
0.50	70	263,168	26.0837	26.10	-0.02	-0.1
0.50	100	425, 984	18.4923	18.50	-0.01	0.0
0.50	130	648,192	13.2618	13.31	-0.05	-0.4
0.90	70	263,168	27.3177	27.23	0.09	0.3
0.90	100	410,624	19.4685	19.50	-0.03	-0.2
0.90	130	623,616	14.1748	14.16	0.01	0.1

Table 5: Results for the value of European Call options with different times of first dividend payments t_1, strike prices K and time steps. Note that for each trial until the convergence 2^{10} simulations were made instead of just one, which explains why with $t_1 = 0.5$ and $K = 70$ we needed the exact same number of simuations as when $t_1 = 0.9$ and $K = 70$. The tolerance parameters used were $\varepsilon = 0.1$ and $\alpha = 0.1$.

First of all note, that with the Monte Carlo method, we do not increase the number of time steps in the tree or grid to make our results more accurate but instead we increase the number of simulations. We worked with up to 670,000 simulations in order to provide optimal results.

The results we obtain all seem to be quite consistent with the paper's results as can be seen in the absolute difference figures (column "Abs. diff" in Table 5). The highest deviation occurs in the case of strike price $K = 70$ and first dividend date at $t_1 = 0.90$ where our result lies \$0.09 above the option value of $VN1000$. On average our results lie \$0.02 below the paper's value which gives an average percentage difference of -0.01%.

One interesting thing to note is that with the Monte Carlo method, due to its different approach, we do not obtain values which are always smaller than the paper's results (as was the case with the FDM approach and the European option values for the binomial tree approach). Instead, the option values produced by the Monte Carlo method fluctuate around the values calculated by Vellekoop and Nieuwenhuis (2006).

6.4 Numerical Efficiency

Due to the way of working of Monte Carlo method, there is little to be done in order to improve the accuracy of the results but to choose smaller tolerance parameters or implementing more complex pseudo random generators (weather related in most cases). But to improve the speed of calculations we might try doing the following.

1. Try to obtain a theoretical result about the number of simulations needed and set it as a starting number of simulations.
2. Analyze if it is worth to simulate more paths every trial the loop does.
3. Use different algorithms for every option; in ours both, call and put, are expected to converge in order to stop the simulations.
4. Simulate only *cum* and *ex* dividend prices instead of the whole path.

The downside of the last point is that it would not work for American options, as well as for path dependent options. It is also this flexibility to replicate the market "*as is*" that makes Monte Carlo method so intuitive and easy to apply for options with a fixed maturity at the cost of a convergence of order $O(\sqrt{n})$.

7 Conclusion

In this paper, we were investigating three different numerical methods for the pricing of European and American options with discrete dividends under the Black Scholes framework. In particular, we used the binomial tree approach, the finite difference method and the Monte Carlo approach to replicate two numerical examples provided by Vellekoop and Nieuwenhuis (2006).

Considering Tables 1-5, we obtain that the binomial approach yields the results closest to the ones obtained by Vellekoop and Nieuwenhuis (2006), for both the American and the European option. For the European option, the Monte Carlo approach gives results which are still very accurate, only slightly more deviating from the paper's result than the binomial ones. Finally, also the FDM approach results in satisfying option values although being furhtest away from the given reference figures. Consequently, in terms of accuracy the binomial tree approach works best under these simplified settings.

Also taking speed of execution as well as difficulty of implementation into consideration, the binomial tree approach seems to beat its competitors. Note however that these two factors are also influenced by the programmers' abilities and the power of the computer used.

Only taking into account our two simplified examples, one could come to the conclusion that the binomial tree approach is the best among the three models introduced. However, in order to use this model in real life, one would have to include some adaptations like e.g. jumps. In this case, the binomial tree approach ceases to work so well and it is a lot easier to implement changes in the FDM or the Monte Carlo method. In fact, this is the reason why the binomial method is rarely used in practice.

In conclusion, we are of the opinion that the binomial tree approach works best under simplified circumstances while in real life the other two methods should be preferred.

References

A. Quarteroni, F. Saleri, P. G. (2010). *Scientific Computing with Matlab and Octave.* springer.

Gilli, M. and E. Schumann (2009). Implementing binomial trees. Technical report.

Glasserman, P. (2004). *Monte Carlo methods in financial engineering*, Volume 53. Springer.

Haug, E. G., J. Haug, and A. Lewis (2003). Back to basics: a new approach to the discrete dividend problem. *Wilmott magazine 9*, 37–47.

Miller, M. H. and F. Modigliani (1961). Dividend policy, growth, and the valuation of shares. *the Journal of Business 34*(4), 411–433.

Vellekoop, M. and J. Nieuwenhuis (2006). Efficient pricing of derivatives on assets with discrete dividends. *Applied Mathematical Finance 13*(3), 265–284.

W. H. Press, S. A. Teukolsky, W. T. V. . B. P. F. (1992). *Numerical Recipes in C.* Cambridge University Press.

Wilmott, P. (2007). *Paul Wilmott on Quantitative Finance, 3 Volume Set.* Wiley. com.